Non-perturbative saddle point for the effective action of disordered and interacting electrons in 2D

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We find a non-perturbative saddle-point solution for the non-linear sigma model proposed by Finkelstein for interacting and disordered electronic systems. Spin rotation symmetry, present in the original saddle point solution, is spontaneously broken at one-loop, as in the Coleman-Weinberg mechanism. The new solution is singular in both the disorder and triplet interaction strengths, and it also explicitly demonstrates that a non-trivial ferromagnetic state appears in a theory where the disorder average is carried out from the outset.

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Understanding the combined effects of disorder and interactions in electronic systems has proven to be both an extremely interesting and difficult problem. The issue has regained a new surge of interest since the discovery by Kravchenko and co-workers [1] of a possibly conducting state in two-dimensional (2-D) Si-MOSFETs (see also Ref. [2]). Although there has been intense debate within the theoretical community as on the origin of the transition [3], little has been accomplished that is in as solid grounds as the scaling theory of localization for the non-interacting problem [4].

The initial attempt to establish a scaling theory for the interacting problem was put forward by Finkelstein [5], who studied an extension of Wegner's non-linear sigma model containing singlet and triplet interaction couplings. The renormalization group (RG) flow takes the interaction coupling constants to strong coupling, away from the perturbative starting point of a diffusive Fermi liquid state; hence no conclusive picture has emerged from this approach. Among the outstanding theoretical issues is the nature of the magnetic state signaled by the divergence of the triplet interaction.

Let us begin by briefly discussing the breakdown of the RG flow for the interaction. The one loop RG equation for γ_2, the ratio between the triplet and the singlet interactions, is given by [5–7]

$$\frac{d\gamma_2}{dl} = \frac{1}{2}\, t\,(1+\gamma_2)^2 \ , \tag{1}$$

where t is the resistance. Although the resistance also gets renormalized, it is instructive to solve this equation for a "fixed" t, which leads to a breakdown at an RG scale $l = \frac{2}{t(1+\gamma_2)}$, or at a length scale $\lambda = \Lambda^{-1}\, e^{\frac{2}{t(1+\gamma_2)}}$ (Λ is a momentum cutoff).

It is very useful to draw an analogy to the problem of BCS superconductivity at this point. Within Shankar's renormalization group approach to fermions [8], one can perturb around the Fermi liquid fixed point, and obtain the flow equation for the BCS interaction $V<0$:

$$\frac{dV}{dl} = -V^2 \ . \tag{2}$$

This equation also breaks down at a finite length scale – the coherence length. At this length scale, the non-perturbative physics of pairing takes over. Similarly, the breakdown of Eq. (1) implies that non-perturbative physics dominates the behavior of the system. Such physics cannot be accessed perturbatively from the diffusive saddle point.

However, the main difference between these two examples is that, in the case of BCS superconductivity, we know what the physics of the non-perturbative fixed point is from the BCS mean-field solution. Shankar's RG procedure allows us to find that there is an instability of the Fermi liquid, but it alone does not access the non-pertubative BCS solution. The situation is completely analogous in the diffusive Fermi liquid problem: we know that the non-interacting saddle point is unstable under the RG flow, but we cannot determine the non-perturbative physics solely from the flow.

In this paper, we find a non-perturbative self-consistent solution for the saddle point of the interacting and disordered electronic problem by looking at the one-loop effective potential of the non-linear sigma model. The new saddle point spontaneously breaks spin rotation symmetry for *any* value of the triplet coupling constant.

The starting point of our calculation is Finkelstein's Q-matrix model of unitary class [5]. The disorder-averaged N-replica partition function of the interacting problem reads

$$\langle Z_N \rangle = \int DQ\, e^{-S[Q]}, \tag{3}$$

where

$$S[Q] = \frac{\pi\nu_F}{4\tau}\,\mathrm{tr}\,Q^2 - \mathrm{tr}\ln\, G^{-1} + Q\hat{\gamma}Q, \tag{4}$$

with the tensor $\hat{\gamma}$ containing the singlet and triplet interactions, and

$$G^{-1} = i\omega + \left(\frac{\Delta}{2m} + \mu\right) + \frac{i}{2\tau}\, Q. \tag{5}$$

The Q matrix considered here has the following structure: $Q = Q_{ij}^{\alpha\beta}(k-k';\omega,\omega')$, where the pairs ij and $\alpha\beta$ denote replica and spin indices, respectively, while k, k' are momenta and ω, ω' are frequencies. The trace operation runs over all these indices. For our purposes, it will be sufficient to consider only the zero-temperature limit.

Let us expand the action for $Q = Q_0 + \delta Q$:

$$\begin{aligned} S[Q] &= S[Q_0] \\ &\quad + \frac{\pi\nu_F}{2\tau}\,\mathrm{tr}(Q_0\,\delta Q) - \frac{i}{2\tau}\mathrm{tr}(G_0\,\delta Q) + 2\,Q_0\hat{\gamma}\delta Q \\ &\quad + \frac{\pi\nu_F}{4\tau}\,\mathrm{tr}(\delta Q\,\delta Q) - \frac{1}{8\tau^2}\,\mathrm{tr}(G_0\,\delta Q\,G_0\,\delta Q) - \delta Q\hat{\gamma}\delta Q \\ &\quad + \frac{i}{24\tau^3}\,\mathrm{tr}(G_0\,\delta Q\,G_0\,\delta Q\,G_0\,\delta Q), \end{aligned} \tag{6}$$

where G_0^{-1} follows from Eq. (5) after replacing Q by Q_0. The reason for keeping up to order δQ^3 when searching for the saddle point will become transparent below. After this expansion, the usual next step is to choose Q_0 such that the term linear in δQ vanishes; this leads to the saddle point used by Finkelstein. Equivalently, this condition on the linear terms can be cast as

$$\frac{\delta S}{\delta Q}\Big|_{Q_0} = \Gamma_0^{(1)}\Big|_{Q_0} = 0, \tag{7}$$

where $\Gamma_0^{(1)}$ is the tree level (or zero-loop) one-vertex function. By writing the usual saddle-point equation as a condition on the one-vertex $\Gamma_0^{(1)}$ (or linear in δQ) potential, we are approximating the whole vertex function or effective potential $\Gamma[Q] = V[Q]$ by the tree level potential $\Gamma_0[Q] = S[Q]$. This leads to the tree level saddle point ${Q_0}_{ij}^{\alpha\beta}(q;\omega,\omega') = \mathrm{sgn}(\omega)\,\delta_{q,0}\,\delta_{\omega\omega'}\,\delta_{\alpha\beta}\,\delta_{ij}$. Fluctuations are then parametrized by slow unitary rotations $Q = U^{-1}(r)\,Q_0\,U(r)$. The sigma model constrained to this manifold has an instability in the triplet interaction channel, which signals that the wrong saddle point has been chosen.

We will proceed by considering not the tree level potential or action in the derivation of saddle point, but the effective potential generated by the fluctuations. There is a symmetry breaking not present at the tree (or "classical") level, but that surfaces at higher loop order ("quantum") in the potential, an effect know as the Coleman-Weinberg mechanism [9]. With this in mind, the correct saddle for the interacting disorder electron problem should be found at one-loop order:

$$\frac{\delta V}{\delta Q}\Big|_{Q_0} = \Gamma^{(1)}\Big|_{Q_0} \approx \Gamma_0^{(1)}\Big|_{Q_0} + \Gamma_1^{(1)}\Big|_{Q_0} = 0. \tag{8}$$

This is the reason why we kept up to the cubic term in Eq. (6). To one-loop order, the cubic term contributes to $\Gamma_1^{(1)}$, as shown in Fig 1. Notice that, if it were not for the interactions, this contribution would vanish because it is proportional to $N \to 0$ in the zero-replica limit. The interaction keeps the contribution alive because it requires all replicas in the loop to be the same.

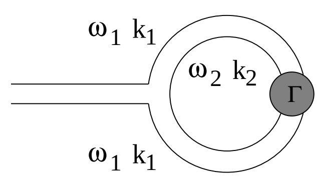

FIG. 1. One-loop contribution to the one-vertex term in the effective potential.

Contracting two of the δQ's from the cubic term, we obtain (spin and replica indices are not shown):

$$\delta V = \frac{i}{8\tau^3} \sum_{\omega_1,\omega_2,\omega_3} \sum_{k_1,k_2,k_3} G_0(\omega_1,k_1)\,G_0(\omega_2,k_2)\,G_0(\omega_3,k_3)\,\langle \delta Q_{\omega_1\omega_2}(k_1-k_2)\,\delta Q_{\omega_2\omega_3}(k_2-k_3)\rangle\,\delta Q_{\omega_3\omega_1}(k_3-k_1) \tag{9}$$

or, alternatively,

$$\Gamma_1^{(1)} = \frac{i}{8\tau^3} \sum_{\omega_2} \sum_{k_1,k_2} \left[G_0(\omega_1,k_1)\right]^2\, G_0(\omega_2,k_2)\,\langle \delta Q_{\omega_1\omega_2}(k_1-k_2)\,\delta Q_{\omega_2\omega_1}(k_2-k_1)\rangle. \tag{10}$$

The propagator of density fluctuations $\mathcal{D} = \langle \delta Q\,\delta Q\rangle$ follows from the quadratic part of Eq. (6) and depends on the saddle Q_0 through G_0. Redefining the frequencies and momenta in the sums, one has

$$\Gamma_1^{(1)} = \frac{i}{8\tau^3} \sum_{\omega'} \sum_{k,q} \left[G_0(\omega,k)\right]^2 G_0(\omega+\omega',k+q)\,\mathcal{D}(\omega,q) \tag{11}$$

Next, we use the above $\Gamma_1^{(1)}$ and find the new saddle solu-

tion of Eq. (8). The solution is a matrix Q_0 homogeneous in space and diagonal in the replica, spin, and frequency indices: ${Q_0}_{ij}^{\alpha\beta}(q;\omega,\omega') = Q_0^\alpha(\omega)\,\delta_{q,0}\,\delta_{\omega\omega'}\,\delta_{\alpha\beta}\,\delta_{ij}$, where we allow for $Q_0^\uparrow(\omega) \neq Q_0^\downarrow(\omega)$. Let us also assume that the electrons interact through a screened (short-ranged) interaction. The new saddle-point equation becomes

$$Q_0^\alpha(\omega) = \frac{i}{\pi\nu_F}\int\frac{d^2k}{(2\pi)^2}\,G_0^\alpha(\omega,k) - \frac{2\tau}{\pi\nu_F}\Gamma_1^{(1)}\Big|_{Q_0} - \frac{4\tau}{\pi\nu_F}\sum_{\alpha,\beta,\mu}\gamma^{\alpha\beta;\mu\mu}\int\frac{d\omega'}{2\pi}\,Q_0^\mu(\omega') \tag{12}$$

The first term on the right-hand side of Eq. (12) is the only one present in the non-interacting case. The second term is the one-loop contribution while the third one corresponds to the Hartree-Fock approximation. In the non-interacting case the solution to the saddle-point equations is simply $Q_0^\alpha(\omega) = \mathrm{sgn}(\omega)$. For the interacting case, we separate the singlet and triplet contributions,

$$\gamma^{\alpha\beta;\mu\nu} = \frac{(\pi\nu_F)^2}{2}\left(\Gamma_s\,\delta_{\alpha\beta}\,\delta_{\mu\nu} + \Gamma_t\sum_{r=1}^{3}\sigma_r^{\alpha\beta}\sigma_r^{\mu\nu}\right), \tag{13}$$

where Γ_s and Γ_t are the singlet and triplet coupling constants, respectively, and σ_r are Pauli matrices. While the singlet channel leads to a shift in the energy band bottom, a repulsive interaction in the triplet channel causes a magnetic instability related to a net spin polarization, similar (but not identical) to the Stoner instability of clean, itinerant electrons in the presence of a ferromagnetic interaction. In order to capture this effect, we make the Ansatz

$$Q_0^\alpha(\omega) = \zeta\,\mathrm{sgn}(\omega) - 2i\tau\Delta_\alpha, \tag{14}$$

where ζ and Δ_α have to be determined self-consistently. For simplicity, we set $\Gamma_s = -\Gamma_t = \Gamma/2$.

The contribution from the triplet channel to the $\mathcal{D} = \langle\delta Q\,\delta Q\rangle$ propagator can be written as

$$\mathcal{D}(\omega,q) = -\frac{\Gamma(\pi\nu_F)^2}{\zeta^2}\,D_0^{\alpha\beta}(\omega,q)\,D_2^{\alpha\beta}(\omega,q) \tag{15}$$

with

$$D_{0,2}^{\alpha\beta}(\omega,q) = \frac{\zeta^2(2/\pi\nu_F)}{z_{0,2}|\omega| + Dq^2 + i\,\mathrm{sgn}(\omega)\Delta_{\alpha\beta}}. \tag{16}$$

Here D is the diffusion constant, $\Delta_{\alpha\beta} = \Delta_\alpha - \Delta_\beta$, $z_0 = z$ (the frequency renormalization factor), and $z_2 = z + \nu_F\Gamma$. It becomes apparent that ζ renormalizes amplitudes, similarly to the wave-function renormalization which appears in the RG treatment of the problem [6,7,10]. By dimensional analysis, we expect the bare interaction coupling constant Γ to be replaced by Γ/ζ^2 – we have already taken that into account when writing Eq. (15). Thus, the contribution to $\Gamma_1^{(1)}$ with opposite spins ($\alpha \neq \beta$) in the loop is

$$\begin{aligned}\Gamma_1^{(1)}\Big|_{Q_0} &\approx \frac{i}{8\tau^3}\int\frac{d^2k}{(2\pi)^2}\int\frac{d^2q}{(2\pi)^2}\int\frac{d\omega'}{2\pi}\frac{1}{[i\omega - \epsilon_k + iQ_0^\alpha(\omega)/2\tau]^2}\,\frac{1}{i(\omega+\omega') - \epsilon_{k+q} + iQ_0^\beta(\omega+\omega')/2\tau} \\ &\quad\times\frac{-4\zeta^2\Gamma}{\left[\bar{z}|\omega'| + Dq^2 + i\,\mathrm{sgn}(\omega')\Delta_{\alpha\beta}\right]^2} \\ &\approx -\frac{\zeta^2\nu_F^2\Gamma}{8\pi g\bar{z}\tau}\frac{\mathrm{sgn}(\omega)}{(\zeta + i\Delta_{\alpha\beta}\tau)^2}\left[-\ln\sqrt{(\omega\tau)^2 + (\Delta_{\alpha\beta}\tau/\bar{z})^2} + i\arctan(\Delta_{\alpha\beta}/\bar{z}\omega)\right],\end{aligned} \tag{17}$$

where $g = \nu_F D$ is the dimensionless conductance in 2-D and $\bar{z} = (z + z_2)/2$. In Eq. (17) we have constrained $|\omega|, |\Delta_{\alpha\beta}| \ll 1/\tau$.

With the one-loop contribution Eq. (17), one can now replace it in Eq. (12) and solve self-consistently for the magnetization bandwidth $\Delta = \Delta_\uparrow - \Delta_\downarrow$. After some algebraic manipulations, we find that the Δ in the saddle-point solution must satisfy

$$\Delta \approx \frac{\bar{z}}{\tau}\exp\left[-\frac{(2\pi\bar{z})^2 g}{2\nu_F\Gamma}\left(\frac{1}{\nu_F\Gamma} - 1\right)\right]. \tag{18}$$

The upper bound in frequency of the diffusion propagator sets the prefactor of the exponential in Eq. (18). As a result, Δ is proportional to the elastic scattering rate, $1/\tau$, rather than the total bandwidth or the Fermi energy.

Equation (18) points to the existence of a non-zero spin polarization $\Delta > 0$ for *any* positive value of Γ, provided that the dimensionless conductance is finite. Only for infinite g we recover the usual Stoner instability characteristic of clean systems, namely, a ferromagnetic instability at $\Gamma > 1/\nu_F$. In the particular case of a *finite* system, the same tendency towards spin polarization was found by Andreev and Kamenev [11]. This suggests that the ferromagnetic instability may be a robust property of 2D disordered interacting electrons, since their starting point was rather different than ours. For the finite system case, they used an exact basis representation for the non-interacting problem, combined with disorder averaged Hartree-Fock matrix elements. These matrix ele-

ments were dressed by the diffusive dynamics to lowest order in $1/g$ and provided a contribution to the magnetization similar, but not identical, to our one-loop calculation with the disorder average carried out from the very beginning.

It has been suggested that the divergence of the triplet coupling is connected to the formation of local magnetic moments in the system [12,13]. Notice, however, that the polarization of the saddle-point solution in Eq. (18) connects continuously with the Stoner instability in the limit of a clean system. This is suggestive of a dirty ferromagnetic state [14] where there exists a residual coupling between the local moments, causing a tendency towards a global ferromagnetic order at $T = 0$. If the localized moments are coupled through, for example, an RKKY interaction that has alternating signs, one would expect not a ferromagnetic state, but rather something similar to a spin glass state [15]. There can be two solutions to this puzzle. One is that the characteristic size of the local moment diverges, and hence the system is in a ferromagnetic state (at $T = 0$) according to the saddle-point solution in this paper. The other alternative is that there may be another saddle, which in contrast to ours, breaks replica symmetry (RS). Let us remind that in spin glass models the replica symmetric solution is unstable in the glass phase, which is characterized by a non-zero Edwards-Anderson order parameter without magnetic order. Our saddle-point solution, on the other hand, finds directly a magnetic order parameter $\Delta \neq 0$. In this case, it appears unlikely that RS breaking may resolve the issue. In addition, any RS breaking solution must be such that it recovers the usual Stoner instability in the clean limit.

Another important issue to consider is the divergence of the frequency rescaling factor z in the RG equations for the sigma model expanded around the zero-loop saddle. In the introduction, we claimed that the divergence of the interactions happen at a finite length scale; however, the divergence of z would mean that the associated energy scale goes to zero. The latter conclusion is not correct for the following reason. The coupling constant with the leading divergence is the triplet, and when this coupling leaves the perturbative RG limit, one can no longer trust the flow. The saddle with the non-trivial magnetization resolves the leading triplet divergence, and that should stop the z divergence as well.

We would like to conclude the paper by discussing the issue of localization in disordered and interacting systems. The nature of the triplet channel instability is clearly captured by a disordered averaged effective theory through a non-trivial saddle-point solution with a non-zero magnetization. Moreover, the saddle point provides a natural starting point of an new RG treatment for the localization problem. Once $\Delta \neq 0$ is taken, it appears that the renormalization of the diffusion constant D should proceed similarly to the non-interacting case. Although a renormalization program should be carried out to confirm whether D scales to zero, it appears unlikely that starting from the new saddle will lead to delocalization in 2-D for weak interactions. This, however, would not preclude a correlated (super) conducting state, for example, as proposed by Phillips and coworkers [3].

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